PMAT 435 - ANALYSIS 1

1. Sequences

Definition 1. A sequence (s_n) of real numbers is said to **converge** to the real number s if for each $\varepsilon > 0$, $\exists N$ such that $\forall n \geq N, |s_n - s| < \varepsilon$

Theorem 1. *A convergent sequence has exactly one limit, i.e. the limit is unique.*

Proof. Let $s_n \to a$ and $s_n \to b$ with $a < b$.
Let $\varepsilon = \frac{b-a}{2} > 0$.
$s_n \to a \Rightarrow \exists N$ such that $\forall n \geq N, |s_n - a| < \varepsilon$
$\Rightarrow, \forall n \geq N, a - \varepsilon < s_n < a + \varepsilon = \frac{a+b}{2} = b - \varepsilon$
$\Rightarrow s_n \notin N(b : \varepsilon) \forall n \geq N$.
Thus, s_n does not converge to b □

Theorem 2. *Every convergent sequence is bounded.*

Proof. Let $s_n \to s$. Then, $\exists N$ such that $\forall n \geq N, |s_n - s| < 42$
$\Rightarrow \forall n \geq N, s - 42 < s_n < s + 42$.
Let $a = max\{s_1, s_2, s_3, \cdots, s_N, s - 3\}$ and $b = max\{s_1, s_2, s_3, \cdots, s_N, s + 3\}$
Then, $a \leq s_n \leq b \forall n \Rightarrow (s_n)$ is bounded. □

Theorem 3. *(Limit Theorems) Suppose that* $a_n \to a$*,* $b_n \to b$ *and* $c \in \mathbb{R}$*. Then*

(1) $(a_n \pm b_n) \to a \pm b$
(2) $(a_n \cdot b_n) \to a \cdot b$
(3) $c \cdot (a_n) \to c \cdot a$
(4) *if* $b \neq 0, b_n \neq 0 \, \forall n$*, then* $\frac{a_n}{b_n} \to \frac{a}{b}$

Proof. 1. Let $\varepsilon > 0$.
$a_n \to a \Rightarrow \exists N_1$such that $\forall n \geq N_1, |a_n - a| < \frac{\varepsilon}{2}$.
Similarly, $b_n \to b \Rightarrow \exists N_2$ such that $\forall n \geq N_2, |b_n - b| < \frac{\varepsilon}{2}$.
Let $N = max\{N_1, N_2\}$.
Then, $\forall n \geq N, |a_n - a| < \varepsilon$ and $|b_n - b| < \varepsilon$

$$\begin{aligned} \Rightarrow |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

$\Rightarrow (a_n + b_n) \to (a + b)$.
2. Since (a_n) converges, it is bounded, by M say. Then, $|a_n| \leq M \forall n$.
Let $\varepsilon > 0$
Then $\exists N_1$ such that, $\forall n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{M+|b|+1}$
and $\exists N_2$ such that, $\forall n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{M+|b|+1}$.
Let $N = max\{N_1, N_2\}$. Then, $\forall n \geq N$,

$$\begin{aligned}|a_nb_n - ab| &= |a_nb_n - a_nb + a_nb - ab| \\ &\leq |a_nb_n - a_nb| + |a_nb - ab| \\ &= |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| \\ &< M \cdot \tfrac{\varepsilon}{M+|b|+1} + |b| \cdot \tfrac{\varepsilon}{M+|b|+1} \\ &= \varepsilon \cdot \tfrac{M+|b|}{M+|b|+1} < \varepsilon\end{aligned}$$

□

Theorem 4. ***(Squeeze Theorem)*** *Suppose that $a_n \to s, b_n \to s$ and if .$\forall n,\ a_n \leq c_n \leq b_n$, then $c_n \to s$*

Proof. Let $\varepsilon > 0$.
Then $\exists N_1$ such that, $\forall n \geq N_1,\ |a_n - s| < \varepsilon$
and $\exists N_2$ such that, $\forall n \geq N_2,\ |b_n - s| < \varepsilon$.
Let $N = max\,\{N_1, N_2\}$. Then, $\forall n \geq N$,
$s - \varepsilon < a_n < s + \varepsilon$ and $s - \varepsilon < b_n < s + \varepsilon$
$\Rightarrow s - \varepsilon < a_n \leq c_n \leq b_n < s + \varepsilon$
$\Rightarrow s - \varepsilon < c_n < s + \varepsilon$
$\Rightarrow c_n \to s$ □

Definition 2. (s_n) is said to **diverge** to $+\infty$ if $\forall M > 0, \exists N$ such that, $\forall n \geq N,\ s_n \geq M$

Definition 3. (s_n) is said to **diverge** to $-\infty$ if $\forall M > 0, \exists N$ such that $\forall n \geq N,\ s_n \leq -M$

Definition 4. (s_n) is **monotonic increasing** if, $\forall n,\ s_n \leq s_{n+1}$

Definition 5. (s_n) is **strictly increasing** if, $\forall n,\ s_n < s_{n+1}$

Theorem 5. *Every monotonic increasing sequence that is bounded above converges. (Similarly, every monotonic decreasing sequence that is bounded below converges.)*

Proof. Since s_n is bounded above, by the Completeness Axiom, $s = sup\{s_n | n \in \mathbb{N}\}$ exists.
We want to show that $s_n \to s$
For each $\varepsilon > 0, s - \varepsilon$ is not an upper bound for $\{s_n | n \in \mathbb{N}\}$
Thus, $\exists N$ such that $s_N > s - \varepsilon$.
Since s_n is increasing, $\forall n \geq N, s_n \geq s_N > s - \varepsilon$
Since s is an upper bound for s_n, $\forall n, s_n \leq s$
Then, $\forall n \geq N, s - \varepsilon < s_n < s < s + \varepsilon$
$\Rightarrow \forall n \geq N, |s_n - s| < \varepsilon$
$\Rightarrow s_n \to s$

□

Theorem 6. *Every monotonic decreasing sequence that is bounded below converges.*

Proof. 1. Model proof on the previous proof.
2. (Alternative method)
Let (s_n) be monotonic decreasing and bounded below, by m say.
Define (t_n) by $t_n = -s_n \forall n$
Then, $\forall n, s_n \geq s_{n+1} \Rightarrow -s_n \leq -s_{n+1} \Rightarrow t_n \leq t_{n+1}$ so (t_n) is monotonic increasing.
Also, $\forall n, m \leq s_n \Rightarrow -m \geq -s_n \Rightarrow -m \geq t_n$
Thus, (t_n) is bounded above and monotonic increasing. By the Monotonic Convergenge Theorem, (t_n) converges. □

Theorem 7. *If (s_n) is monotonic increasing but not bounded above, then* $s_n \to \infty$

Proof. Since s_n is not bounded above, for each $M > 0, \exists N$ such that $s_N > M$
Also, (s_n) is monotonic increasing so, $\forall n \geq N, s_n \geq s_N > M$
Therefore $s_n \to \infty$ □

Theorem 8. *If (s_n) is monotonic decreasing but not bounded below, then* $s_n \to -\infty$

Definition 6. Suppose that $(s_n)_{n\in\mathbb{N}}$ is a sequence. A **subsequence** of (s_n) is a sequence of the form $(s_{n_k}) = s_{n_1}, s_{n_2}, s_{n_3}, \cdots$. Note that $n_1 < n_2 < n_3 < \cdots$ so that n_k increases as k increases.

Theorem 9. *If $s_n \to s$, then $s_{n_k} \to s$ for all subsequences (s_{n_k}) of (s_n).*

Proof. For each $\varepsilon > 0, \exists N$ such that $\forall n \geq N, |s_n - s| < \varepsilon$
Then, $\forall k \geq N, n_k \geq n_N \geq N \Rightarrow |s_{n_k} - s| < \varepsilon$ and $(s_{n_k}) \to s$ □

Theorem 10. *Every sequence has a monotonic subsequence.*

Proof. Call a term s_n **dominant** if $s_n > s_m \forall m > n$
Case 1: There are finitely many dominant terms
Let n_1 be such that there are no dominant terms after or including s_{n_1}
Then $\exists n_2 > n_1$ such that $s_{n_2} > s_{n_1}$
But, since $n_2 > n_1, s_{n_2}$ is not a dominant term
Then $\exists n_3 > n_2$ such that $s_{n_3} > s_{n_2} > s_{n_1}$
Continuing with this construction, $\exists n_1 < n_2 < n_3 < n_4 < \cdots$ such that $s_{n_1} < s_{n_2} < s_{n_3} < s_{n_4} < \cdots$
Therefore, (s_{n_k}) is a monotonically increasing subsequence of (s_n)
Case 2: There are infinitely many dominant terms
Let $s_{n_1}, s_{n_2}, s_{n_3}, \cdots$ be the subsequence of dominant terms of (s_n)
Then, since s_{n_1} is a dominant term and since $n_1 < n_2$ we have $s_{n_1} > s_{n_2}$
But, since s_{n_2} is a dominant term and since $n_2 < n_3$ we have $s_{n_2} > s_{n_3}$
In general, s_{n_k} is a dominant term and $n_k < n_{k+1} \Rightarrow s_{n_k} > s_{n_{k+1}}$
Therefore, (s_{n_k}) is a decreasing subsequence of (s_n) □

Theorem 11. *(Bolzano-Weierstrass) Every bounded sequence has a convergent subsequence.*

Proof. Let (s_n) be a bounded sequence. Then it has a monotonic subsequence (s_{n_k}). Since (s_n) is bounded, it follows that (s_{n_k}) is also bounded. By the monotonic convergence theorem (s_{n_k}) converges. □

Definition 7. A sequence (s_n) is said to be a **Cauchy sequence** if for each $\varepsilon > 0, \exists N$ such that $\forall m, n \geq N$, $|s_m - s_n| < \varepsilon$

Theorem 12. *Every Cauchy sequence is bounded.*

Proof. Let (s_n) be a Cauchy sequence.
Then, $\exists N$ such that $\forall m, n \geq N, |s_m - s_n| < 42$
In particular, $\forall n \geq N, |s_N - s_n| < 42$
$\Rightarrow \forall n \geq N, s_N - 42 < s_n < s_N + 42$
Let $a = min\{s_1, s_2, \cdots, s_N, s_N - 42\}$ and $b = max\{s_1, s_2, \cdots, s_N, s_N + 42\}$
Then, $\forall n, a \leq s_n \leq b$ so (s_n) is bounded □

Theorem 13. *Every convergent sequence is a Cauchy sequence.*

Proof. Let (s_n) be a convergent sequence
For each $\varepsilon > 0, \exists N$ such that $\forall n \geq N, |s_n - s| < \frac{\varepsilon}{2}$
Then, $\forall m, n \geq N, |s_n - s| < \frac{\varepsilon}{2}$ and $|s_m - s| < \frac{\varepsilon}{2}$
Therefore, $\forall m, n \geq N$,

$$\begin{aligned} |s_m - s_n| &= |s_m - s + s - s_n| \\ &\leq |s_m - s| + |s_n - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, (s_n) is a Cauchy sequence

□

Theorem 14. ***(Cauchy criterion for convergence)*** *Every Cauchy sequence converges.*

Proof. Let (s_n) be a Cauchy sequence.
Then, (s_n) is bounded and (Bolzano-Weierstrass) has a convergent subsequence, (s_{n_k}) say.
Let $s_{n_k} \to s$.
Claim: $s_n \to s$
Let $\varepsilon > 0$.
Then $\exists N_1$ such that $\forall m, n \geq N_1, |s_m - s_n| < \frac{\varepsilon}{2}$
Also, $\exists K$ such that $\forall k \geq K, |s_{n_k} - s| < \frac{\varepsilon}{2}$
Let $N = max\{N_1, K\}$, and let $k = \lceil N \rceil$ (note: we can pick any k such that $k \geq N$)
Then $\forall n \geq N$,

$$\begin{aligned} |s_n - s| &= |s_n - s_{n_k} + s_{n_k} - s| \\ &\leq |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

That is, $s_n \to s$

□

Definition 8. $\limsup s_n = \lim_{n\to\infty} \sup\{s_m : m > n\}$

Definition 9. $\liminf s_n = \lim_{n\to\infty} \inf\{s_m : m > n\}$

Note. If $u_n =$inf $\{s_m : m > n\}$ and $v_n =$sup$\{s_m : m > n\}$, then

(1) $u_n \leq s_m \leq v_n \; \forall m > n$
(2) $u_n \leq$ inf $\{s_m : m > n\} = u_{n+1}$, so (u_n) is monotone increasing
(3) $v_n \geq$ sup $\{s_m : m > n\} = v_{n+1}$, so (v_n) is monotone decreasing
(4) (v_m) in monotonic decreasing, (u_m) is monotonic increasing.
(5) $u_m \leq v_n \; \forall m, \forall n$. Thus (u_n) is bounded above by each and every v_n, (v_n) is bounded below by each and every u_n. Thus, they are convergent.

Definition 10. (s_n) is said to be **eventually** in a set A if $\exists N$ such that $\forall n \geq N, s_n \in A$

Definition 11. (s_n) is said to be **frequently** in a set A if $\forall N$, $\exists n \geq N$ such that $s_n \varepsilon A$. That is, (s_n) has a subsequence lying entirely in A.

Theorem 15. ***(Characterisation Theorem** for* $\limsup s_n$ *) Let* (s_n) *be a bounded sequence. The following statements are equivalent:*

(1) $s = \limsup \; s_n$
(2) (s_n) *is eventually* $<$ *all numbers that are* $> S$ *and frequently* $>$ *all numbers that are* $< S$

Proof. ($\Longrightarrow$) Assume that $s = \limsup\ s_n$
$s = \lim_{n\to\infty} v_n$ where (v_n) is monotonic increasing
$\Rightarrow\ s = inf\{v_n : n \geq 1\}$
Let $M > s$. Then M is not a lower bound for $\{v_n : n \geq 1\}$
Then, $\exists N$ such that $v_N = sup\{s_m : m > n\} < M$
Therefore $\forall m \geq N, s_m \leq v_N \leq M$
Next, let $m < S$. We want to show that (s_n) is frequently $> m$.
Suppose the contrary, so (s_n) is not frequently $> m$.
Therefore, (s_n) is eventually $\leq m$
That is, $\exists N$ such that $\forall n \geq N, s_n \leq m$
Therefore m is an upper bound for $\{s_n : n > N\}$
Therefore $m \geq \sup\{s_n : n > N\} = v_n \geq s$, which is a contradiction since $m < s$
($\Longleftarrow$) Let s be a number with the two properties mentioned in 2) above.
Let $t = \limsup\ s_n$. By ($\Longrightarrow$) above, t has the two properties mentioned in 2)
If $s < t$, consider $s < \frac{s+t}{2} < t$.
But s satisfying 2) $\Rightarrow s_n < \frac{s+t}{2}$ eventually and t satisfying 2)$\Rightarrow s_n > \frac{s+t}{2}$ eventually, which is a contradiction.
Similarly, if $t < s$ we get a contradiction.
Thus $t = s$

□

Theorem 16. *Let (s_n) be a bounded sequence. The following statements are equivalent:*

(1) $s = \limsup s_n$
(2) (s_n) *is eventually $>$ all numbers that are $< S$ and frequently $<$ all numbers that are $> S$*

Theorem 17. *Let (s_n) be a bounded sequence. Then,*

(1) $\lim_{n\to\infty} s_n = S \Rightarrow \limsup\ s_n = \liminf s_n = S$
(2) $\limsup\ s_n = \liminf\ s_n = s \Rightarrow \lim_{n\to\infty} s_n = s$

Definition 12. Let (s_n) be a **bounded** sequence. S is called a **subsequential limit** of (s_n) if it is the limit of some subsequence of (s_n).

Theorem 18. *Let (s_n) be a bounded sequence. Let S be the set of all subsequential limits of (s_n). Then S is bounded and non-empty. Furthermore, sup $S = \limsup s_n$ and inf $S = \liminf\ s_n$.*

Proof. (s_n) is bounded and, by Bolzano-Weierstrass, has a convergent subsequence, and thus a subsequential limit.
$\Rightarrow\ S \neq \emptyset$ as required.
(s_n) is bounded so $\exists a, b$ such that $\forall n, a \leq s_n \leq b \Rightarrow a \leq s_{n_k} \leq b$ for all subsequences (s_{n_k})
$\Rightarrow\ a \leq t \leq b$ for all subsequential limits t, so S is bounded.
We now show that $\sup S = \limsup\ s_n$
Let $M = \sup\ S$ and $s = \limsup\ s_n$
If $s < M$, then s_n will be eventually $< \frac{s+M}{2}$
Therefore, if (s_{n_k}) is any subsequence of (s_n), s_{n_k} will be eventually $< \frac{s+M}{2}$
$\Rightarrow\ \forall t \in S, t \leq \frac{s+M}{2}$ so $\frac{s+M}{2}$ is an upper bound for S
$\Rightarrow\ \frac{s+M}{2} \geq \sup\ S = M$ which is a contradiction.
If $s > M$, then $s > \frac{s+M}{2} > M \Rightarrow s_n$ is frequently $> \frac{s+M}{2}$
$\Rightarrow$ there exists a subsequence (s_{n_k}) of (s_n) such that $s_{n_k} > \frac{s+M}{2} \forall k$
(s_n) is bounded$\Rightarrow (s_{n_k})$ is bounded, so (s_{n_k}) has a convergent subsequence $(s_{n_{k_l}})$.
But then $(s_{n_{k_l}})$ is also a subsequence of (s_n)
Let $s_{n_{k_l}} \to t$.
Since $s_{n_{k_l}} > \frac{s+M}{2} \forall l$ and since t is a subsequential limit, we have $t \geq \frac{s+M}{2} > M$, a contradiction.
Thus, $s = M$

□

Theorem 19. *Let (s_n) be a bounded sequence. Let S be the set of all subsequential limits of (s_n). If $t_n \in S$ and if $t_n \to t$, then $t \in S$.*

Proof. Let t_1 be a subsequential limit of (s_n), i.e. $t_1 \in S$. Then $\exists n_1$ such that $|s_{n_1} - t_1| < 1$
Let $t_2 \in S$. Then there exists a subsequence of (s_n) that has limit t_2
$\Rightarrow \exists n_2 (> n_1)$ such that $|s_{n_2} - t_2| < \frac{1}{2}$
Similarly, $\exists n_3 (> n_2 > n_1)$ such that $|s_{n_3} - t_3| < \frac{1}{3}$
Continuing thus, we can find $n_1 < n_2 < n_3 < n_4 < \cdots < n_k < \cdots$ such that $|s_{n_k} - t_k| < \frac{1}{k}$. Then,

$$\begin{aligned} |s_{n_k} - t| &= |s_{n_k} - t_k + t_k - t| \\ &\leq |s_{n_k} - t_k| + |t_k - t| \\ &< \frac{1}{k} + |t_k - t| \to 0 \; as \; k \to \infty \end{aligned}$$

Therefore, $s_{n_k} \to t$ and so $t \in S$ □

Theorem 20. *Let (s_n) be a bounded sequence. Let S be the set of all subsequential limits of (s_n). Then,* $\limsup s_n \in S$ *and* $\liminf s_n \in S$

Proof. Let $t = \limsup s_n = \sup S$
$\Rightarrow \forall n \in \mathbb{N}, t - \frac{1}{n}$ is not an upper bound for S
$\Rightarrow \exists t_n \in S$ such that $t - \frac{1}{n} < t_n$
But $t_n \in S \Rightarrow t_n \leq t$. That is, $t - \frac{1}{n} < t_n < t$
$\Rightarrow -\frac{1}{n} < t_n - t < 0 < \frac{1}{n}$
$\Rightarrow |t_n - t| < \frac{1}{n} \to 0$
$\Rightarrow t_n \to t$
$\Rightarrow t \in S$

□