CPSC 491 - LECTURE NOTES SUMMARY

1. Math Review

If you're up to speed with your calculus, the following should be fairly straighforward. But if you haven't looked at this since Math 251 or 221 in first year and you feel intimidated by the concepts or the notation, it's worth spending some time becoming reacquainted with the following topics. This section is aimed, primarily, at those of you who have rusty math skills; if you can get fairly comfortable with the material covered here, the rest of the course will be **much much easier**. I shall go into more detail in this section than in any later ones (if I have time for any more!) since these are the tools which are most critical. With a solid background established here, the course should not provide too many problems. If concepts are still unclear, **let me know!**- Dave.

1.1. Calculus.

1.1.1. *Sequences.*

A **sequence** is (formally) described as a function from the natural numbers $1, 2, 3, \ldots$ into the reals, i.e. $f : \mathbb{N} \rightarrow \mathbb{R}$. What does this mean? With each number $1.2.3. \ldots$ is associated a real number $f(1), f(2), f(3), \ldots$ respectively. However, it is more usual (and convenient) to describe a sequence value at n as x_n, rather than $f(n)$.

Less formally, we can view a sequence (or infinite sequence) as an ordered list having a first element (or term) but no last element. It is usual, but not necessary, that a sequence starts with element x_1. Both $x_0, x_1, x_2, x_3, \ldots$ and $x_{42}, x_{43}, x_{44}, x_{45}, \ldots$ are valid sequences.

We write $\{x_n\}_{n=1}^{\infty} = x_1, x_2, x_3, \ldots$

Examples of sequences :

1. Let $x_n = n$. Then $\{x_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty} = 1, 2, 3, 4, 5, 6, \ldots$
2. Let $x_n = (-1)^{n+1}$. Then $\{x_n\}_{n=1}^{\infty} = \left\{(-1)^{n+1}\right\}_{n=1}^{\infty} = 1. -1, 1, -1, \ldots$
3. Let $x_n = \frac{1}{n}$. Then $\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$
4. Sequences can be defined recursively. Let $x_1 = 1$, $x_2 = 1$, $x_{n+2} = x_n + x_{n+1}$. Then $\{x_n\}_{n=1}^{\infty} = 1, 1, 2, 3, 5, 8, 13, \ldots$ (This is the Fibonacci sequence.)

We have an intuitive idea about the convergence of a sequence. For instance, it seems reasonable that $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ converges to 0. As n gets larger $\frac{1}{n}$ gets smaller and closer to 0. (Indeed, 0 is the limit of $1. \frac{1}{2}, \frac{1}{3}, \ldots$ - see below for proof.)

Unfortunately, the definition of convergence given below is (unless you are a pure mathematician) pretty ugly, and doesn't appear to help with our intuitive understanding of convergence.

Definition. $\{x_n\}$ is a **convergent sequence** and converges to the real number L if:

for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that, for all $n > N, |x_n - L| < \epsilon$

Remark. This definition, given in class, is different from the $(\epsilon - \delta)$ definition given in the text. The two definitions are equivalent, although the proof of this is not entirely trivial.

Remark. I have no memory of having to apply this definition when I did 491. So it may not be too serious if the following seems totally unfathomable. However, if you don't get it, but would like to, I'll be happy to go over it - Dave

So, what is the definition saying?

Consider, first, the absolute value part of the definition, $|x_n - L| < \epsilon$.

Remember that:

$$\begin{aligned} |a| &= \begin{cases} a & if\ a \geq 0 \\ -a & if\ a < 0 \end{cases} \\ &= max\{a, -a\} \\ &\geq both\ a\ and\ -a \end{aligned}$$

It follows directly that:

$$\begin{array}{llll} & x_n - L \leq |x_n - L| < \epsilon & and & -(x_n - L) \leq |x_n - L| < \epsilon \\ \Rightarrow & x_n < L + \epsilon & and & L - \epsilon < x_n \end{array}$$

Thus, $|x_n - L| < \epsilon \Rightarrow L - \epsilon < x_n < L + \epsilon$

Rephrasing the definition of convergence, then:

for any $\epsilon > 0$ (however small) there is some point (indicated by N) after which all the sequence terms lie within distance ϵ of limit L. Thus, if $x_n \to L$, the terms in the "tail" of the sequence are arbitrarily close to limit L, since $L - \epsilon < x_n < L + \epsilon$ has to hold for any choice of ϵ.

A useful result: If $|x| < 1$, then $x_n \to 0$ (i.e. $\lim_{n\to\infty} x_n = 0$)

Example. Show that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0

Proof. Notice that $\left\{\frac{1}{n}\right\}$ is a decreasing sequence (since $\frac{1}{2} > \frac{1}{3}, \frac{1}{3} > \frac{1}{4}, \ldots, \frac{1}{n} > \frac{1}{n+1}, \ldots$) and notice that all the $\frac{1}{n}$ are positive so that $|x_n| = x_n$.

Let $\epsilon > 0$ and choose $N = \frac{1}{\epsilon}$. (With practice, the choice of N becomes more obvious!)

Then, for all $n > N = \frac{1}{\epsilon}$, $|x_n - 0| = |x_n| = x_n < x_N = \frac{1}{N} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$

Thus, for all $n > N$, $|x_n - L| < \epsilon$, so $\frac{1}{n} \to 0$ as required. □

1.1.2. *Infinite Series.*

(Those of you who have completed Cpsc 413 should be fairly comfortable with this material)

Whereas a sequence is an infinite list of terms, an infinite series is a **sum** of infinitely many terms. $\sum_{n=1}^{\infty} a_k$ is shorthand for the infinite series $a_1 + a_2 + a_3 + \ldots + a_k + \ldots$ We are mainly interested in whether an infinite series converges to some real number S (in which case we write $\sum_{n=1}^{\infty} a_k = S$) or whether it diverges (to $+\infty$ or to $-\infty$). In the case of a convergent series, we can occasionally find the value of S.

$\sum_{n=1}^{m} a_k = a_1 + a_2 + a_3 + \ldots + a_m$ is called a **partial sum**.

To investigate whether or not an infinite series converges we look at the **sequence**, $\{s_n\}$, of partial sums, defined as follows:

$$\begin{array}{lcllcl} s_1 & = & a_1 & & = & \sum_{k=1}^{1} a_k \\ s_2 & = & a_1 + a_2 & & = & \sum_{k=1}^{2} a_k \\ s_3 & = & a_1 + a_2 + a_3 & & = & \sum_{k=1}^{3} a_k \\ \vdots & & \vdots & & & \vdots \\ s_n & = & a_1 + a_2 + a_3 + \cdots + a_n & & = & \sum_{k=1}^{n} a_k \\ \vdots & & \vdots & & & \vdots \end{array}$$

The infinite series $\sum_{k=1}^{\infty} a_k$ is said to converge if and only the sequence of partial sums, $\{s_n\}$, converges to some real number S. Then $\sum_{k=1}^{\infty} a_k = S$.

A series of the form $\sum_{k=1}^{\infty} cr^n$, where c and r are constants, is called a **geometric series**. If this series converges (this depends on the value of r), it is one of the few infinite series that we can sum.

Theorem. $\sum_{n=0}^{\infty} cr^n$ *converges if* $|r| < 1$ *and diverges if* $|r| > 1$. *If the geometric series converges,* $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$

Note. If $|r| = 1$, then $\sum_{k=1}^{\infty} cr^k = \sum_{k=1}^{\infty} c$ or $\sum_{k=1}^{\infty} cr^k = \sum_{k=1}^{\infty} c(-1)^k$ both of which converge if and only if $c = 0$.

Proof. Let $|r| \neq 1$ and consider the partial sum $s_n = \sum_{k=1}^{n} cr^k$. Then,

$$\begin{array}{llcl} & s_n & = & c + cr + cr^2 + cr^3 + \cdots + cr^n \\ \Rightarrow & rs_n & = & cr + cr^2 + cr^3 + \cdots + cr^{n+1} \\ \Rightarrow & s_n - rs_n & = & c - cr^{n+1} \\ \Rightarrow & s_n & = & \frac{c(1-r^{n+1})}{1-r} \end{array}$$

The infinite series converges if and only if the sequence of partial sums, $\{s_n\}$, converges. So, we are interested in the convergence of $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{c(1-r^{n+1})}{1-r}$ which converges to $\frac{c}{1-r}$ if and only if $|r| < 1$.

Then, $\sum_{n=0}^{\infty} cr^n$ converges $\Leftrightarrow s_n$ coverges $\Leftrightarrow |r| < 1$. □

1.1.3. *Taylor Series and Taylor's Theorem.*

A **power series** (about $x = x_0$) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

where the a_n are called the (constant) coefficients of the power series.

Example. Let's look at the geometric series again. We learnt above that $\sum_{n=1}^{\infty} cr^n$ converges for $|r| < 1$. Thus,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \; for\ all \; |x| < 1$$

We say that $1 + x + x^2 + x^3 + \cdots$ is a **power series representation** of the function $f(x) = \frac{1}{1-x}$ for all $|x| < 1$. Now we are viewing x as a variable instead of r as some constant. Consider $x = \frac{1}{2}$. Then, we get the following result:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = f\left(\frac{1}{2}\right) = \frac{1}{1-\frac{1}{2}} = 2$$

(Note that this power series representation is only valid for $|x| < 1$.)

Taylor Series. If a function $f(x)$ is infinitely differentiable at $x = x_0$, (that is, $f^{(k)}(x_0)$ exists for $k = 1, 2, 3, 4, 5, \ldots$) then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots$$

is called the **Taylor series of f about $x = x_0$.** (A Taylor series expansion around $x_0 = 0$ is usually referred to as a Maclaurin series; a Maclaurin series is just a specialised case of a Taylor series and has no special properties.) **If the series converges, then the power representation is equal to** $f(x)$. That is,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots$$

Example. Find the Taylor series about 0 (the Maclaurin series) of $f(x) = sin(x)$.

$$\begin{array}{lclclcl}
f'(x) & = & cos(x) & \Rightarrow & f'(0) & = & 1 \\
f''(x) & = & -sin(x) & \Rightarrow & f''(0) & = & 0 \\
f^{(3)}(x) & = & -cos(x) & \Rightarrow & f^{(3)}(0) & = & -1 \\
f^{(4)}(x) & = & sin(x) & \Rightarrow & f^{(4)}(0) & = & 0 \\
f^{(5)}(x) & = & cos(x) & \Rightarrow & f^{(5)}(0) & = & 1 \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

Then, the Taylor series expansion for $sin(x)$ is

$$\begin{aligned}
sin(x) &= f(0) + f'(0)(x-0) + \tfrac{f''(0)}{2!}(x-0)^2 + \tfrac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots \\
&= 0 + 1 \cdot x + 0 + \tfrac{(-1)x^3}{3!} + 0 + \tfrac{1 \cdot x^5}{5!} + \cdots \\
&= x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \cdots
\end{aligned}$$

Strictly speaking, we should not equate the Taylor series sum to the function $f(x) = sin(x)$ until we have verified that the series converges. Well, it does converge. We won't cover the tests for convergence of series in this course (if you're interested, take Math 349) so you can take my word for it! The series converges for all values of x so the equality holds for all x.

Partial sums of Taylor series can be used to approximate functions. For example, the Taylor series expansion for e^x about $x_0 = 0$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \; for\ all\ x$$

(this can be verified by using the same procedure that we used above to find the Taylor series expansion for $cos(x)$) But, to calculate accurately the value of e^x we need to evaluate infinitely many terms - a good retirement project for someone, but hardly practical. We can use finitely many terms and get an approximate value for e^x. For a convenient example, choose $x = 1$. Then,

# of terms			appromate value for e
1	e	$\approx$	1
2	e	$\approx$	2
3	e	$\approx$	2.5
4	e	$\approx$	2.666666667
5	e	$\approx$	2.708333667

This "appears" to be converging to some value (my guess is 2.7182818) but an approximation is useless without some idea of the size of the possible error in that approximation, and appearances can be deceptive. (The sequence of partial sums $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \cdots =$ $1, 1.5, 1.8333, 2.0833, 2.2833, 2.4999, 2.5929, \cdots$ also gives some appearance of convergence yet this sequence diverges to $+\infty$.) Taylor's Theorem will solve this problem of estimating the error size for us.

Let $P_n(x)$ be the partial sum of the first $(n+1)$ terms of a Taylor series expansion for $f(x)$. This is called the **Taylor polynomial of order** n. The remaining terms in the Taylor series expansion is the error, $R_n(x)$, in our calculation of $f(x)$, caused by truncating the series and ignoring the "tail" of the series.

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} + \cdots}_{R_n(x)}$$

Theorem. *(Taylor's Theorem)*

If the (n+1)th derivative of f exists on an interval containing x and x_0 and $f(x) = P_n(x) + R_n(x)$, then the remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

for some c between x and x_0. ($R_n(x)$ is known as the Lagrange Remainder)

This doesn't directly give the exact value of the error; we still have to choose a c that maximises $R_n(x)$. This is not generally a trivial task and in CPSC 491 it is unlikely that you will have to calculate this.